Directed Surfaces in Disordered Media

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The critical exponents for a class of one-dimensional models of interface depinning in disordered media can be calculated through a mapping onto directed percolation. In higher dimensions these models give rise to directed surfaces, which do not belong to the directed percolation universality class. We formulate a scaling theory of directed surfaces, and calculate critical exponents numerically, using a cellular automaton that locates the directed surfaces without making reference to the dynamics of the underlying interface growth models.

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The last decade's intense theoretical, numerical, and experimental interest in the growth and roughening of interfaces has been fueled in part by the interdisciplinary aspects of the subject [1]. Applications include fluid-fluid displacement [2], imbibition in porous media [3], and the motion of flux lines in superconductors [4]. In such systems, an interface advances through a disordered (typically porous) medium, under the influence of an external driving force, F. There exists a critical value, F_c, of this force, such that for $F < F_c$ the interface is pinned by the disorder, while for $F > F_c$ it moves with a constant velocity v. In the vicinity of the depinning transition ($F = F_c$), $v \sim f^\theta$, where $f = (F - F_c)/F_c$. The correlation length (ξ) characterizing the size of the pinned regions in the plane of the interface diverges at F_c as $\xi \sim f^{-\nu}$. For $F > F_c$ the width of the interface in steady state varies with the system size L as $w(L) \sim L^\alpha$. Here θ, ν, and α are the velocity, correlation length, and roughness exponents, respectively.

In this paper we consider "directed percolation depinning" (DPD) models [3,5], which are believed to describe the depinning transitions in a variety of systems, among them interfaces described by the Kardar-Parisi-Zhang equation [6] with quenched noise [7] and imbibition experiments [3]. The DPD model is defined on a d-dimensional lattice [8], with periodic boundary conditions in the $d - 1$ interface dimensions and open boundary conditions in the dimension corresponding to the direction of motion of the interface (the z direction, say). Sites are randomly occupied by impurities with probability p, and a fluid is imagined to push its way upward from below. At each time we randomly choose one of the impurity-free "dry" sites neighboring the interface that separates the wet and dry regions. The interface advances by "wetting" the chosen site, and *any site below it in the same column* (along the z direction) [3]. In this model only continuous, unbroken surfaces of impurities that cover the entire $(d - 1)$-dimensional cross section of the lattice *and contain no overhangs* can successfully block the interface [3,5]. In 2D, this blocking surface is a collection of 1D lines, which can be viewed as the backbone [9] of the infinite cluster of a $(1 + 1)$-dimensional DP problem that starts from one side of the lattice and percolates towards the other. Thus the scaling exponents (α, θ, ν) for the DPD model in 2D can be obtained from the mapping to the $(1 + 1)$-dimensional DP problem [3,5]. In particular, the correlation length of DPD can be identified with the longitudinal correlation length of DP, giving $\nu = \nu_\parallel^{\mathrm{DP}} \approx 1.73$. Similarly, one obtains the exponents α and θ as $\alpha = \nu_\perp^{\mathrm{DP}}/\nu_\parallel^{\mathrm{DP}} \approx 0.63$ and $\theta = \nu_\parallel^{\mathrm{DP}} - \nu_\perp^{\mathrm{DP}} \approx 0.63$, where $\nu_\perp^{\mathrm{DP}} \approx 1.10$ is the transverse correlation length exponent of DP.

While the DP theory correctly predicts all relevant exponents for 2D DPD models, the mapping fails in higher dimensions. There is a simple topological reason for this: In $(1 + 1)$ dimensions, the directed percolation backbone is a collection of 1D *lines*, capable of blocking the motion of the one-dimensional interface in 2D DPD. In 3D, however, only an unbroken 2D *surface* of pinning sites without overhangs can block the advance of the 2D interface. The collection of such surfaces forming the backbone of a blocking cluster in 2D (or in any other dimension) is referred to as a *directed surface* [10]. Numerical measurements of the critical exponents characterizing the depinning transition of DPD models for $d > 2$ [11] confirm that directed surfaces belong to a universality class different from DP.

Here we take several steps toward a description of the DPD depinning transition for $d > 2$ in terms of directed surfaces. First we introduce a deterministic two-state cellular automaton (CA) that finds the directed surface for a system with an arbitrary distribution of impurities, and in arbitrary dimension. The CA produces this surface without making reference to the dynamics of the underlying DPD interface growth models. Focusing on the directed surfaces allows us to formulate a scaling theory of the transition, and thus describe this problem using the standard formalism of critical phenomena. Moreover, the CA allows us to measure numerically exponents not available from the DPD models. Using the

derived scaling laws we thereby obtain a complete set of scaling exponents characterizing the static properties of directed surfaces.

Cellular automata and directed surfaces.—We first define the CA rules, and then explain why they generate the desired directed surfaces. Consider a square 2D lattice in the (x, z) plane. At time $t = 0$ each site (x, z) is either independently occupied by an impurity $[s_0(x, z) = 1]$ with probability p or is empty $[s_0(x, z) = 0]$. The CA rule is defined as $s_{t+1}(x, z) = 1$ if the following three conditions are simultaneously satisfied: $s_t(x, z) = 1$; $s_t(x - 1, z - 1) + s_t(x - 1, z) + s_t(x - 1, z + 1) > 0$; $s_t(x + 1, z - 1) + s_t(x + 1, z) + s_t(x + 1, z + 1) > 0$. If any of these are not satisfied, then $s_{t+1}(x, z) = 0$. Boundary conditions are periodic in x and open in z. The CA rule leaves untouched any occupied site that locally belongs to an unbroken 1D path roughly perpendicular to the z direction. If an occupied site (x, z) has an occupied neighbor or next near neighbor in both the $(x - 1)$st and $(x + 1)$st columns [see Fig. 1(a)], then it is part of such a path. However, if the occupied site is at the tip of a dangling branch [e.g., the site B in Fig. 1(a)], then at the next time step it will be removed, since it is missing a neighbor in one of the adjacent columns. Once B is removed, however, A is left without a neighbor in one adjacent column, and so is itself removed at the following time step. In this fashion, all dangling branches and isolated clusters of impurity sites are systematically eliminated. Thus under successive applications of the rule, all impurities that do *not* belong to the backbone of the infinite cluster of the equivalent $(1 + 1)$-dimensional DP problem [e.g., the shaded path in Fig. 1(a)], wherein z and x, respectively, correspond to space and time, ultimately disappear. If such an infinite DP cluster does not exist, then the fixed point of the CA has all sites empty. Figure 1(b) shows the result of applying this rule in a system of size $L = 100$ with $p = 0.55$.

The generalization to higher dimensions is straightforward. We discuss only the 3D case, defined on a cubic lattice with axes labeled (x, y, z); the extension to $d > 3$ is obvious. In 3D, $s_{t+1}(x, y, z) = 1$ if the following five conditions are simultaneously satisfied: $s_t(x, y, z) = 1$; $s_t(x - 1, y, z - 1) + s_t(x - 1, y, z) + s_t(x - 1, y, z + 1) > 0$; $s_t(x + 1, y, z - 1) + s_t(x + 1, y, z) + s_t(x + 1, y, z + 1) > 0$; $s_t(x, y - 1, z - 1) + s_t(x, y - 1, z) + s_t(x, y - 1, z + 1) > 0$; $s_t(x, y + 1, z - 1) + s_t(x, y + 1, z) + s_t(x, y + 1, z + 1) > 0$. If at least one of these conditions fails, then $s_{t+1}(x, y, z) = 0$. Periodic and open boundary conditions apply in the (x, y) and z directions, respectively.

As in 2D, for $d \geq 3$ the CA rule removes any site that does not belong to a locally continuous $(d - 1)$-dimensional surface perpendicular to the z direction. It retains only unbroken surfaces, without overhangs, capable of blocking the advance of an interface in the associated DPD model. It therefore locates *exactly*

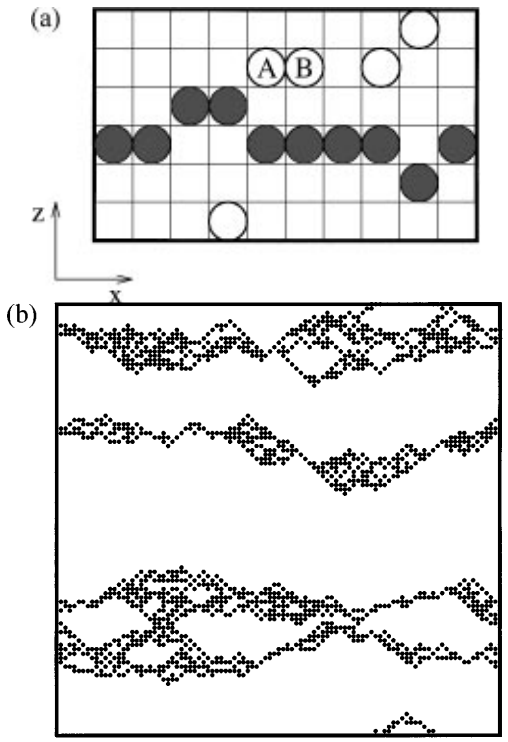

FIG. 1. (a) Schematic representation of the action of the CA rule. The final state of the CA contains only the backbone of the underlying DP cluster (solid circles). Open circles represent dangling branches or isolated clusters that will eventually be eliminated. (b) Directed surfaces produced by the CA rule in 2D with $p = 0.55$ and $L = 100$.

all directed surfaces in arbitrary dimension. Thus one expects the critical exponents measured for the DPD model and for the directed surfaces generated by the CA to coincide. This conclusion is supported by the numerical results presented below.

Scaling exponents for directed surfaces.—The final state (fixed point) of the CA depends on p. For small p there are no directed surfaces in the system, so the average density of the final state, $\rho(p) = 1/L^d \sum_{\mathbf{r}} s(\mathbf{r})$, where $\mathbf{r} = (i_1, i_2, \ldots, i_d)$ is zero. Increasing p, one reaches a critical value p_c so that, for $p > p_c$, $\rho(p)$ is nonzero. In analogy with percolation, one expects that $\rho(p) \sim (p - p_c)^{\beta}$. The exponent β is not known, and has not previously been measured numerically, except in 2D, where the directed surface is the backbone of the infinite cluster of $(1 + 1)$-dimensional DP. Since the exponent β for this backbone is known to be $2\beta^{\mathrm{DP}}$ [12], where $\beta^{\mathrm{DP}} \sim 0.28$ in $(1 + 1)$ dimensions, we have $\beta \approx 0.56$ in 2D.

To characterize further the directed surfaces for arbitrary d, it is helpful to define parallel (to the average orientation of the surface) and perpendicular correlation lengths $\xi_{\parallel}$ and $\xi_{\perp}$, respectively. Near the depinning transition they diverge as $\xi_{\parallel} \sim (p - p_c)^{-\nu_{\parallel}}$ and $\xi_{\perp} \sim (p - p_c)^{-\nu_{\perp}}$.

Finally, one can define exponents associated with the distribution of the sizes of voids or holes in the directed

surface, i.e., empty regions totally surrounded by surface sites [11,13]. The probability distribution functions $P_v(v)$ for the void volumes v and $P_i(s)$ for linear void sizes s in the ith direction are expected to behave algebraically: $P_v(v) \sim v^{-\tau_v}$, and, in 3D, e.g., $P_{x,y}(s) \sim s^{-\tau_\parallel}$ and $P_z(s) \sim s^{-\tau_\perp}$. As recent work by Huber, Jensen, and Sneppen shows [14], in 2D the void exponents τ_v, $\tau_\parallel$, and $\tau_\perp$ can be related to the exponents characterizing the size distribution of avalanches associated with the dynamics of the DPD model [11] or its self-organized version, the "self-organized depinning" model [13,15].

Scaling theory.—The first advantage of describing directed surfaces in static terms similar to those fruitfully employed for the percolation problem is that we can use the standard scaling arguments familiar from critical phenomena to characterize the correlations and the fractal nature of those surfaces [16]. As a first step, we derive a set of scaling relations for the critical exponents of directed surfaces. Imagine a coarse-grained density field $\psi(\vec{x})$ for the surface. Under rescaling of distances in the directions parallel and perpendicular to the surface via $x_\parallel = bx'_\parallel$ and $x_\perp = b^\alpha x'_\perp$, the field ψ and the distance $\Delta \equiv p - p_c$ from the critical point are assumed to rescale according to $\psi(x_\parallel, x_\perp) = b^\chi \psi'(x'_\parallel, x'_\perp)$ and $\Delta = b^{-1/\{\zeta\}}\Delta'$, respectively. Here b (> 1) is the length rescaling factor, and α, ζ, and χ are critical exponents.

The rescaling of lengths implies that the correlation length exponents are given by $\nu_\parallel = \zeta$ and $\nu_\perp = \alpha\zeta$: Denoting the average density of the surface $\langle\psi(x)\rangle$ by M, it follows that M vanishes like Δ^β as $\Delta \to 0^+$, with $\beta = -\chi\nu_\parallel$.

To express χ in terms of the exponents for other correlation functions, note that the scaling relations above imply

$$G(\vec{x}, \Delta) = \Delta^{-2\chi\nu_\parallel} h(x_\parallel \Delta^{\nu_\parallel}, x_\perp \Delta^{\nu_\perp}), \qquad (1)$$

where $G(\vec{x}, \Delta) \equiv \langle\psi(\vec{x})\psi(\vec{0})\rangle$. For $\Delta > 0$, G has two parts, a disconnected piece equal to M^2 and a connected piece $G_c(\vec{x}, \Delta) \equiv G(\vec{x}, \Delta) - M^2$. $G(\vec{x}, \Delta)$ is proportional to the steady-state probability that both sites $\vec{0}$ and $\vec{x}$ belong to the directed surface, i.e., to the product of the probability of site $\vec{0}$ belonging to the directed surface and the conditional probability that $\vec{x}$ belongs to the directed surface, given that $\vec{0}$ does. This makes it clear that $G(\vec{x}, \Delta)$ vanishes such as M as $\Delta \to 0$, whereupon close to p_c, $G_c(x_{\parallel,\perp}) \sim M x_{\parallel,\perp}^{-\beta/\nu_{\parallel,\perp}}$. Using the standard notation of critical phenomena, we define the exponents $\eta_\parallel$ and $\eta_\perp$ via $G_c(x_{\parallel,\perp}) \sim M x_{\parallel,\perp}^{-(d-2+\eta_{\parallel,\perp})}$, leading to the scaling laws [17]

$$\beta = \nu_{\parallel,\perp}(d - 2 + \eta_{\parallel,\perp}). \qquad (2)$$

The exponents $\eta_\parallel$ and $\eta_\perp$ are straightforwardly related to the fractal dimensions [18] $D_\parallel$ and $D_\perp$ of the directed surface in the $\parallel$ and $\perp$ directions by [19] $D_\parallel = 1 - \eta_\parallel$ and $D_\perp = 3 - d - \eta_\perp$. Moreover, the overall fractal dimension D, defined by the total number of surface points within a distance R of a given point on the surface growing like R^D, is $D = 2 - \eta_\perp = d - \beta/\nu_\perp$.

Finally, for voids, we have the scaling relations $\tau_\parallel = 1 + (\tau_v - 1)(d - 1 + \alpha)$ and $\tau_\perp = 1 + (\tau_v - 1) \times (d - 1 + \alpha)/\alpha$. In 2D the void exponents can be related to $\nu_\perp$, $\nu_\parallel$, and β through the formula [14] $\tau_v - 1 = (\nu_\parallel + \nu_\perp - 2\beta)/(\nu_\parallel + \nu_\perp)$.

Numerical results.—In 2D all the exponents defined here are available from the DP analogy. For higher d, $\nu_\parallel$, $\nu_\perp$, and hence α have been obtained from DPD simulations [11]. Others, such as β, $\eta_{\parallel,\perp}$, and $D_{\parallel,\perp}$, are difficult to compute from DPD, and so have not been measured. The CA representation makes numerical determination of these rather straightforward. Moreover, Eq. (2) shows that, given $\nu_\parallel$ and $\nu_\perp$, one additional independent exponent suffices to fix all the others, except the void exponents, which for $d > 2$ have so far not been related to the others.

In principle, the most straightforward exponent to measure numerically is β, obtained from plotting the density of the final state of the CA as a function of p. In 2D the data show a fairly unambiguous scaling regime, yielding the value $\beta \approx 0.5$, consistent with the known value $\beta^{\mathrm{BB}} = 2\beta^{\mathrm{DP}} \approx 0.56$. In 3D near p_c, ρ shows a rather precipitous jump that sharpens with increasing sample size, suggestive of either a first order phase transition or a very small exponent β. The data for $\rho(p)$ are insufficient for one to choose between these possibilities. We therefore resort to determining β by measuring $D_\parallel$ and $D_\perp$, and using the scaling relations $D_\parallel = d - 1 - \beta/\nu_\parallel$ and $D_\perp = 1 - \beta/\nu_\perp$, and the values $\nu_\parallel = 1.18 \pm 0.10$ and $\nu_\perp = 0.57 \pm 0.05$, taken from DPD simulations [11]. The values for β thus obtained, together with our values of $D_\parallel$ and $D_\perp$, are listed in Table I. Our numerical values for the void exponents are also given. The existence of a fairly clear scaling regime for $D_\parallel$ with a value less than 2 argues against the occurrence of a first-order phase transition (as, of course, does the continuous phase transition observed in the 3D DPD model). The scaling observed in the data of Fig. 2(b) (spanning, for P_v, more than 2 orders of magnitude in void sizes) also strongly supports the inference of

TABLE I. Exponents obtained from numerical simulations using the CA rule, and from scaling relations.

Exponent	2D	3D
β	0.5 ± 0.07	0.1 ± 0.02
$D_\parallel$	0.71 ± 0.03	1.9 ± 0.05
$D_\perp$	0.54 ± 0.05	0.8 ± 0.1
τ_v	1.7 ± 0.1	2.2 ± 0.1
τ_x	2.2 ± 0.1	3.9 ± 0.2
τ_z	2.8 ± 0.2	7.0 ± 0.4

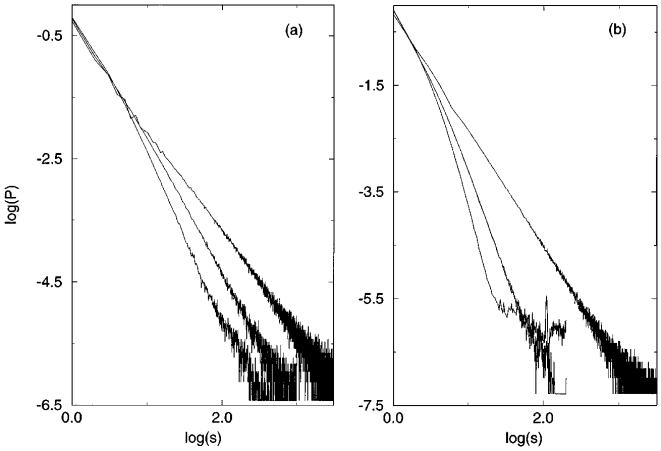

FIG. 2. Void size distributions determined numerically using the CA rule. (a) 2D: The upper, middle, and lower curves correspond to $P_v(v)$, $P_x(s)$, and $P_z(s)$, respectively. We used $p = 0.54$, $L = 1000 \times 1000$, and averaged over 250 runs. (b) 3D: The upper and lower curves correspond to $P_v(v)$ and $P_z(s)$, respectively. In the middle we have two curves superposed, corresponding to $P_{x,y}(s)$. We used $p = 0.74$, $L = 100 \times 100 \times 100$, and averaged over 1500 runs.

a second-order transition. The 2D value for τ_v is consistent with the prediction $\tau_v = 1.80$ by Huber, Jensen, and Sneppen [14]. Using the 3D numerical values, we estimate from the scaling relations above the value $\alpha \approx 0.48$, in good agreement with the value obtained from the DPD model. This provides further confirmation that the directed surface generated by our CA indeed belongs in the same universality class as DPD.

One of the major benefits of the CA approach introduced here is that it replaces the dynamic models used to study the properties of directed surfaces with a static picture. Dynamic models cannot capture all directed surfaces existing in the system. Models with random updating, such as DPD, remove small patches of the directed surface in an uncontrolled fashion during large avalanches. The SOD models [13] also systematically eliminate branches of directed surfaces. By contrast, the CA is the first algorithm that identifies *all* underlying directed surfaces, which in turn determine static exponents such as $\nu_\parallel$, $\nu_\perp$, and α, some of which (such as β) are not accessible from dynamic models. The CA also allows direct calculation of the void-size exponents, from which, at least in 2D, avalanche exponents of the dynamic models can be derived [14].

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[17] More generally, $G_c(\vec{x}, \Delta) \sim M x_\parallel^{-(d-2+\eta_\parallel)} f(x_\perp / x_\parallel^\alpha)$, as $\Delta \to 0$, where the scaling function $f(s) \to 0$ as $s \to 0$, and $f(s) \to s^{-(d-2+\eta_\parallel)/\alpha}$ as $s \to \infty$. It follows that the "susceptibility" $S(\Delta) \equiv \int d^d x G_c(\vec{x}, \Delta)$ behaves as $\Delta^{\nu_\parallel(d-3+2\eta_\parallel-\alpha)}$, whereupon the exponent γ defined by $S(\Delta) \sim \Delta^{-\gamma}$ is given by $\gamma = \nu_\parallel(3 + \alpha - 2\eta_\parallel - d)$.

[18] At $p = p_c$ the average number of points on the surface contained in a $(d-1)$-dimensional hypersphere of radius R, centered at an arbitrary point on the surface and lying in the $(d-1)$-dimensional parallel subspace, is expected to grow like $R^{D_\parallel}$ for large R. An analogous definition holds for $D_\perp$.

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