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LETTER TO THE EDITOR

Multi-affine model for the velocity distribution in fully turbulent flows

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Abstract. A simple multi-affine model for the velocity distribution in fully developed turbulent flows is introduced to capture the essential features of the underlying geometry of the velocity field. We show that in this model the various relevant quantities characterizing different aspects of turbulence can be readily calculated. A simultaneous good agreement is found with the available experimental data for the velocity structure functions, the D_q spectra obtained from studies of the velocity derivatives, and the exponent describing the scaling of the spectrum of the kinetic energy fluctuations. Our results are obtained analytically assuming a single free parameter. The fractal dimension of the region where the dominating contribution to dissipation comes from is estimated to be $D \simeq 2.88$.

Recent studies of the highly complex nature of fully developed turbulence have revealed that the scaling properties of fluid flows at large Reynolds numbers can successfully be described in terms of fractal geometry and multifractal distributions [1–12]. The multifractal description of the singular behaviour of the velocity structure functions and the dissipation field represents the most general approach to turbulence and the existing models can be looked at as various realizations of the associated formalism. In this picture, places with a given singularity are distributed on fractal subsets of dimension depending on the strength of the singularity. In this respect the K41 theory [13] leading to the $E(k) \sim k^{-5/3}$ law for the wavenumber power spectrum of the kinetic energy corresponds to a homogeneously (non-fractally) distributed dissipation field, while in the more recent theories the intermittent nature of turbulence is taken into account through fractal geometry.

Since the direct solution of the Navier–Stokes equations is not feasible for the full velocity field, a traditional and useful way of gaining insight into the geometry of turbulent flows has been the construction of models making analytic treatment possible. In the so-called β model [5] (stimulated partly by the ideas of Mandelbrot on the fractal aspects of turbulence [3]) the kinetic energy is transferred only to a fraction of the smaller eddies in a fractally homogeneous manner, and the dissipation is entirely confined to a fractal support with a dimension D less than 3. However, the experimental results are much more consistent with the recent multifractal models: the random β model [1] is in good agreement with the measurements of higher-order velocity structure functions, while the multifractal spectra derived from the p model [2] are in accord with the experimental results for the generalized dimensions [6] determined from measurements of the dissipation field [2]. In addition, very recent results for the power spectrum of the temperature fluctuations have also indicated that the experimental data could be better fitted using a multifractal analysis [11]. In the random β model the fractal dimension of the set of points in which the velocity has a given type of singularity is considered, while the p model describes the cascade of kinetic energy in the inertial range down to the dissipation level. The central quantities on which the formalism is based in these approaches are nowhere continuous fractal measures related to the velocity distribution or to the concentration of passive scalars. On the other hand, these latter, directly observable physical quantities are much better behaving continuous functions.

Here we propose a simple approach in which the full velocity distribution is directly modelled by a continuous multi-affine function and, in this way, a complete picture capturing the essential features of the geometry underlying turbulent flows is provided. It will be shown that in this approach the relevant quantities can be readily calculated and comparisons can be made with the available experimental data on fully developed turbulence. In particular, we find good agreement with the measured velocity structure functions, the $D(q)$ spectra obtained from studies of the velocity derivatives, and the exponent describing the scaling of the spectrum of the kinetic energy fluctuations. From the fit to the experimental data we obtain an estimate $D \simeq 2.88$ for the fractal dimension of the region where the dominating contribution to dissipation comes from.

As the Navier–Stokes equations in the small viscosity limit ($\nu \to 0$) are invariant under the scaling transformations $\boldsymbol{r} \to \lambda \boldsymbol{r}$, $\boldsymbol{v} \to \lambda^H \boldsymbol{v}(t \to \lambda^{1-H} t$, $\lambda > 0)$ we are led to assume that the velocity distribution satisfies the usual criterion of self-affinity [4] $u(x) \sim \lambda^{-H} u(\lambda x)$, where in the last expression we only considered the behaviour of the x component of the velocity vector. This expression is equivalent to the scaling of the velocity differences of the form $\langle |u(x+\Delta x) - u(x)| \rangle \sim \Delta x^H$. According to the multifractal picture of turbulent flows the scaling exponent H depends continuously on the position in space [1]. This is the property which we call multi-affinity, since it corresponds to an invariance under affine transformations with an infinite hierarchy of rescaling factors depending on the local value of H.

Let us consider the iterative process shown in figure 1 which is perhaps the simplest realization of a self-affine function satisfying the above outlined criteria. This procedure represents a generalization of the construction proposed by Mandelbrot [14] and recently developed further by us [15, 16] to incorporate multiscaling. In each step of the recursion the intervals obtained in the previous step are replaced with the properly rescaled version of the generator which has the form of an asymmetrical z made of three intervals (figure 1(a)). During the procedure every interval which is to be replaced is regarded as a diagonal of a rectangle becoming more and more elongated as the number of iterations k increases.

When going to the next level, one can rescale either (a) the generator shown in figure 1 or (b) its image inverted around the centre of the rectangle. Then the stochastic nature of the turbulent signals can be taken into account by randomly choosing between the possibilities (a) and (b). The function $u(x)$ generated in this way becomes multi-affine in the $k \to \infty$ limit. Our main statement is that the behaviour (geometry) of the one-dimensional projection of the velocity field in the inertial range can be well described by such a multi-affine function. The visual appearance of our model after $k=5$ iterations is compared in figure 2 with a measured time series of the velocity in a turbulent pipe flow behind a grid.

Using the above described velocity distribution we can explicitly calculate for our model the standard quantities used to characterize fully developed turbulent flows. In such flows the qth-order longitudinal velocity structure function $C_q(x)$ is expected to

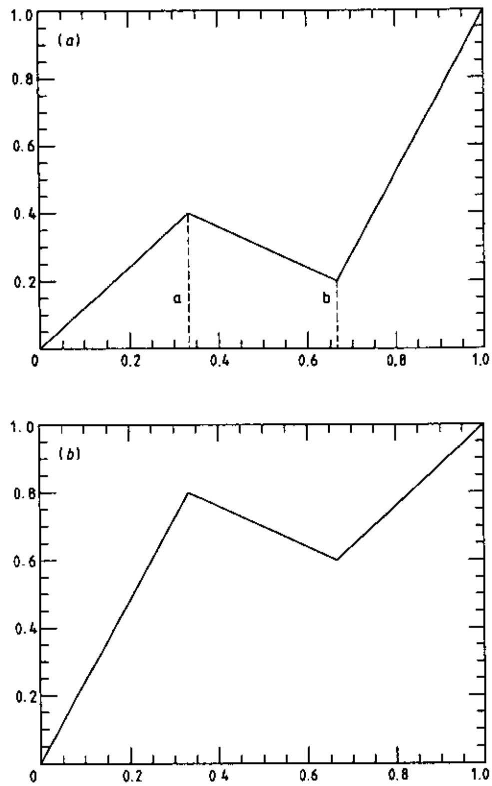

Figure 1. The generators of the multi-affine model shown in figure 2. In each step of the recursion one of the asymmetrical z shaped generators made of three intervals shown in (a) and (b) is picked randomly and used to replace the single intervals obtained in the previous step (for more details see text).

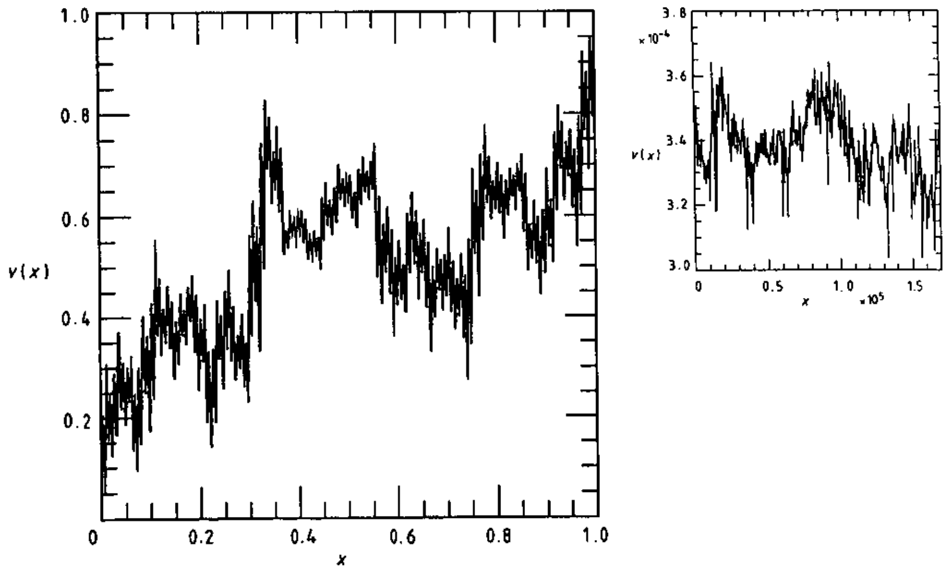

Figure 2. The geometry of the multi-affine model ($k = 5$) for the one-dimensional projection of the velocity field. This figure is for the case $a = 0.67$. The inset shows a typical velocity signal observed in a turbulent pipe flow behind a grid (courtesy of W Goldburg). The equivalence of the spatial and temporal dependences is assumed in accord with the Taylor hypothesis (see e.g. [20]).

scale as

$$C_q(x) = \langle |u(x'+x) - u(x')|^q \rangle \sim x^{qH_q} \tag{1}$$

where the exponent $qH_q = \xi_q$ is one of the major quantities studied in the theory of turbulence. There is a firm theoretical result [17] $C_3(r) \sim r$ from which $H_3 = \frac{1}{3}$ follows. We calculate $C_q(x)$ for our model assuming that the scaling properties are entirely determined by the jumps Δu of the function over intervals of length $\Delta x_k = 3^{-k}$. Thus,

$$C_q(\Delta x_k) = \sum_{m_1=0}^{k} \sum_{m_2=0}^{m_1} \frac{1}{3^k} \binom{k}{m_1} \binom{m_1}{m_2} a^{m_1 q} (a-b)^{(m_1-m_2)q} (1-b)^{(k-m_1)q}. \tag{2}$$

Since (2) can be written as $C_q(x_k) = [a^q + (a-b)^q + (1-b)^q]^k / 3^k$ we have

$$\xi_q = qH_q = \frac{\ln[(a^q + [a-b(a)]^q + [1-b(a)]^q)/3]}{\ln\frac{1}{3}}. \tag{3}$$

In (3) we used $b(a)$ to express the fact that the condition $H_q = \frac{1}{3}$ has to be satisfied and this, through the corresponding equality $a^3 + [a-b(a)]^3 + [1-b(a)]^3 = 1$, reduces the number of independent parameters to one.

The experimental values of Anselmet *et al* [18] for ξ_q can be used to find the best value for the parameter a. Figure 3 demonstrates that the prediction of (3) for $a = 0.67$ provides a good fit to the experimental results. At this point our multi-affine model has become completely determined and the next important step is to check whether the model provides such a good agreement when other quantities determined in independent experiments are considered.

Thus, we turn to calculating the generalized dimensions D_q associated with the qth moment of the normalized distribution of various quantities related to the nth power of the velocity differences between close points. The motivation for this is that, for example, the energy transfer or dissipation rate on scale l is widely accepted to be proportional to $(\Delta u)^3/l$, while the local viscous dissipation rate can be shown to be proportional to $(\partial u/\partial x)^2 \simeq (\Delta u)^2/(\Delta x)^2$. The fractal measure is constructed for our

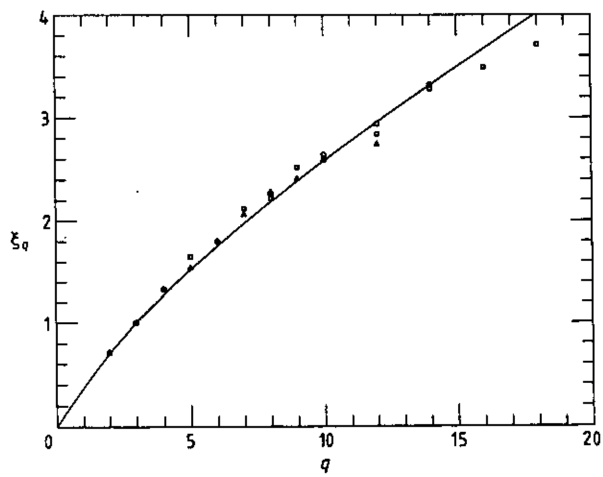

Figure 3. Comparison of the experimentally determined values of the exponent $\xi_q = qH_q$ with the prediction of our model for $a = 0.67$ (continuous line). The experimental data are from [17].

model by defining the normalized distribution $p_i = (\Delta u_i)^n / \Sigma\, (\Delta u_i)^n$, where p_i is the 'probability' or weight associated with the ith interval of length Δx. Then the qth moment of the measure is expected to scale as [6]

$$\chi_q(\Delta x) = \sum_N p_i^q \sim \Delta x^{(q-1)D_q} \tag{4}$$

where the summation is taken over $N = 1/\Delta x$ intervals of length Δx. Equation (4) can be evaluated by noting that the sums of the form $1/N\, \Sigma_N\, (\Delta u_i)^{nq}$ behave as the nqth-order structure functions. Thus,

$$\chi_q(\Delta x) = \frac{\Sigma_N\, (\Delta u_i)^{nq}}{(\Sigma_N\, (\Delta u_j)^n)^q} \sim N^{(1-q)} \frac{C_{nq}(\Delta x)}{(C_n(\lambda x))^q} = \Delta x^{q-1+nq(H_{nq}-H_n)}. \tag{5}$$

From (4) and (5) we get for the generalized dimensions

$$D_q = 1 + \frac{nq}{q-1}(H_{nq} - H_n). \tag{6}$$

The special case of this expression for $n = 3$ was obtained in [19] using different arguments. Here we followed the technique developed in [16]. The predictions of expression (6) can be compared with the experimental results of Meneveau and Sreenivasan [2] for the energy dissipation rate. Assuming that $(\Delta u)^3/l$ describes the dissipation rate down to the level dominated by the viscous forces [5], we obtain the D_q spectrum of the dissipation rate from (6). Inserting $n = 3$ and the corresponding H_{3q} and H_3 values calculated for $a = 0.67$ giving the best fit to the structure function data we find excellent agreement for $q > -2$ (deviations no more than a few per cent). The difference is about 4–6% in the region where the experimental uncertainties are larger so that our results are within the error bars. If we use the square of the first derivative in the expression for the dissipation rate, a very good agreement is found as well (figure 4) for H_{2q} and H_2 values corresponding to $a = 0.6$. This value is somewhat

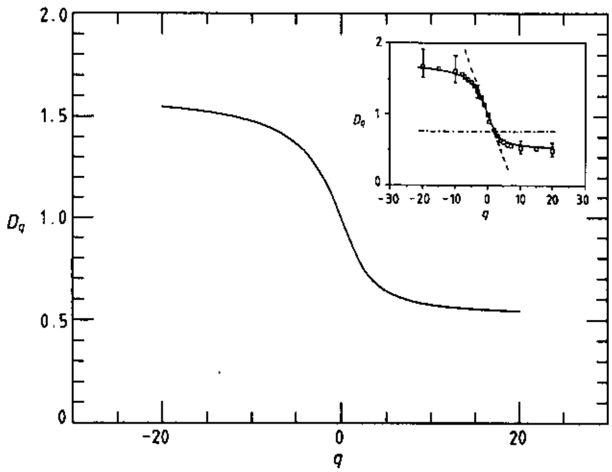

Figure 4. The generalized dimensions D_q associated with the dissipation rate. This spectrum is obtained if the second power of the first derivative of the velocity distribution is used to determine the D_q spectrum. For comparison, the inset shows the related experimental and theoretical (full line) results of Meneveau and Sreenivasan [2].

smaller than the one which was used to find the best fit to the other experiment. On the other hand, the agreement is still good, but less pronounced (deviations are on the order of 10–20%) when $a = 0.67$, is used (the same as for figure 3).

From the D_q spectrum we can obtain an estimate for the fractal dimension D of the set of points at which the essential part of the dissipation takes place in a three-dimensional turbulent flow. According to the standard formalism of multifractals [7–9] we assume that the dominant contribution to the total dissipation $\chi_1(x)$ comes from a fractal subset of dimension f_1 of regions of size δx in which the dissipation is proportional to $(\delta x)^{\alpha_q}$. Then from the well known relation $(q-1)D_q = q\alpha_q - f_q$ it follows that $f_1 = D_1$, and using (3) and (6) we obtain the estimate

$$D = 2 + D_1 \simeq 2.88 \tag{7}$$

where we have taken into account that the fractal dimension of a one-dimensional cut of a D-dimensional object embedded into three dimensions is equal to $D-2$. This is consistent with the related earlier theoretical considerations [1] and the experimental results of [10] which gave estimates close to $D = 2.9$.

The spectrum of the energy fluctuations can readily be obtained in our model as well, assuming that the kinetic energy of an eddy of radius x is proportional to $[\Delta u(x)]^2$. Then the average energy of eddies of size x can be expressed as

$$E(x) \sim \frac{1}{N} \sum_i^N (\Delta u_i)^2 \sim c_2(x) \sim x^{2H_2}. \tag{8}$$

After Fourier transformation we get

$$E(k) = \int x^{2H_2}\, \mathrm{e}^{ikx}\, \mathrm{d}x = \text{constant} \times k^{-(1+2H_2)} \tag{9}$$

where the constant is a k-independent integral remaining after the change of variables $kx = z$. Since for $a = 0.67$, $H_2 \simeq 0.35$ our estimate for the exponent is -1.7 which should be compared with the K41 value $-\frac{5}{3}$. The difference $\delta \simeq 0.033$ is consistent with several model dependent predictions [20] which relate this quantity to the experimentally determined exponent μ describing the correlations of the energy dissipation or to the information dimension D_1 of the dissipation field [5].

Finally, we would like to draw attention to the behaviour of the H_q spectrum close to the region $q = 0$. It is easy to show that in our approach (and in models where the turbulent activity is space filling) $H_{q\to 0} =$ constant, while in models with a fractal support (such as the β and the random β models) $H_{q\to +0} \to \infty$. This observation may provide a basis for the investigations aimed at clarifying which is the more appropriate way of taking into account the intermittent nature of turbulence.

In spirit, our approach can be considered as a combination of the random β and the p models, since our assumptions are for the velocity field (β), but the proposed structure fills the space (p). We have used the simplest multi-affine function to simulate the space (or time) dependence of the velocity and found good agreement with the experimental results concerning various aspects of fully developed turbulent flows. This analytically treatable model allows the calculation of many more quantities, including the scaling functions entering the spectra of such quantities as the energy, the thermal or the velocity fluctuations. Through this, together with the previous and very recently proposed [21, 22] models it may provide further insight into the mechanisms indicating multifractal behaviour in recent experiments [11, 12].

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