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Tracing a diffusion-limited aggregate: Self-affine versus self-similar scaling

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The geometry of diffusion-limited aggregation clusters is mapped into single-valued functions by tracing the surface of the aggregate and recording the X (or Y) coordinate of the position of a walker moving along the perimeter of the cluster as a function of the arc length. Our numerical results and scaling arguments show that the related plots can be considered as self-affine functions whose scaling behavior is determined by the exponent $H=1/D$, where D is the fractal dimension of the aggregates.

I. INTRODUCTION

Numerous recent studies of far-from-equilibrium processes have demonstrated that interface growth phenomena governed by the Laplace equation typically lead to fractal patterns.[1–3] The diffusion-limited aggregation (DLA) model of Witten and Sander[4] represents a simple approach that has been shown to capture the essential features of the structure and development of such interfaces. Since the discovery that DLA clusters are fractals, various methods have been used to characterize their complex spatial behavior. In this paper we use a different approach to provide an alternative way of describing the scaling properties of growing fractal interfaces.

The main idea of our study is that we map the geometry of a DLA cluster into a single-valued function. This is done by tracing the perimeter of an aggregate and plotting the X (or Y) coordinate of the actual position of a walker moving along the perimeter as a function of the distance made by the walker. The plot that is obtained by this procedure can be considered as a length-dependent "signal" and as such it can be treated by a number of methods developed for this purpose. This kind of approach has been used recently to describe the stochastic nature of deterministic growth patterns[5] and to analyze the self-affinity of various curves.[6] Here we intend to describe the scaling properties of plots corresponding to the special type of random walks defined above.

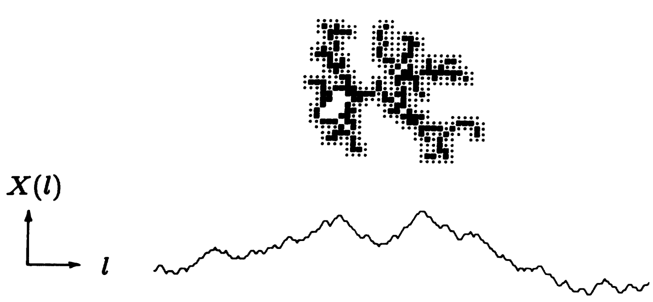

FIG. 1. This figure illustrates the construction of the single-valued function $X(l)$ corresponding to a DLA cluster on the example of a small aggregate.

The actual realization of our approach is carried out as follows: Consider the perimeter of a DLA cluster defined on a lattice, where the perimeter is defined as the set of empty sites that are nearest neighbors to the aggregate. First we fix a direction and a starting point; then we let the walker proceed from one perimeter site to another by not permitting returns to the already visited sites. This process is demonstrated for a small cluster in Fig. 1. It is related to the perimeter walk which has been studied from a different aspect for percolation clusters.[7]

As the walker moves along the perimeter we record its X (or Y) coordinate as a function of the number of steps l. Figure 1(b) shows the corresponding plot. For isotropic clusters (such as an off-lattice diffusion-limited aggregate) there is no difference between the statistical properties of the functions $X(l)$ and $Y(l)$.

In the next section we present the results obtained from simulations carried out on large DLA clusters generated by an off-lattice algorithm. Scaling arguments supporting our numerical results are given in Sec. III. In the last section we discuss the approach described in this paper.

II. SIMULATIONS

To study the scaling properties of the $X(l)$ plots defined above we considered five off-lattice diffusion-limited aggregates each consisting of 250 000 particles. Although the clusters were not grown on a lattice,[8] the coordinates of the particles can be identified with those of the nearest grid points of an underlying square lattice without losing the relevant details of the geometry. The number of steps in the associated perimeter walks was approximately $L=500\,000$.

Since $X(l)$ is apparently rough, it is a natural idea[9] to investigate the scaling behavior of the average of its standard deviation on an interval of length l,

$$\sigma(l)=\left\langle \left[\frac{1}{l}\sum_{i=1}^{l}(X_i-\bar{X})^2\right]^{1/2}\right\rangle , \quad (1)$$

where $\bar{X}$ is the mean value of X on a segment of length l and $\langle\ \rangle$ denotes averaging over many such samples taken from the total signal of L steps in a random or systematic manner. If the width of the function (1) scales with the length of the samples as

$$\sigma(l)\sim l^H , \quad (2)$$

then the plot of the X coordinates can be considered as a self-affine curve[9,10] with a fractal dimension $D_{SA}=2-H$. This fractal dimension is different from that of the DLA cluster itself but, as we shall show later, the two are related. In general, from the scaling behavior of $X(l)$ described by (2) it also follows[10,11] that $X(l)$ satisfies

$$X(l)\simeq\lambda^{-H}X(\lambda l) , \quad (3)$$

which can be considered as an alternative condition for the self-affinity of the recorded signal.

We tested the scaling of $\sigma(l)$ for the traces of DLA clusters by plotting $\ln\sigma$ versus $\ln l$ (Fig. 2), and from the slope of the straight line fitted to the data we found that this scaling behavior is described by an exponent

$$H=0.59\pm0.02 . \quad (4)$$

The above value is remarkably close to $1/D\simeq0.585$, where $D=1.71$ is the fractal dimension of the off-lattice diffusion-limited aggregates in two dimensions.

The self-affine nature of the $X(l)$ plots is demonstrated in Fig. 3, where parts of the original signal are displayed. In accordance with Eq. (3), if one rescales parts of the signal in the horizontal and the vertical directions, respectively, by the factors λ and λ^{-H} the resulting functions remain statistically the same.

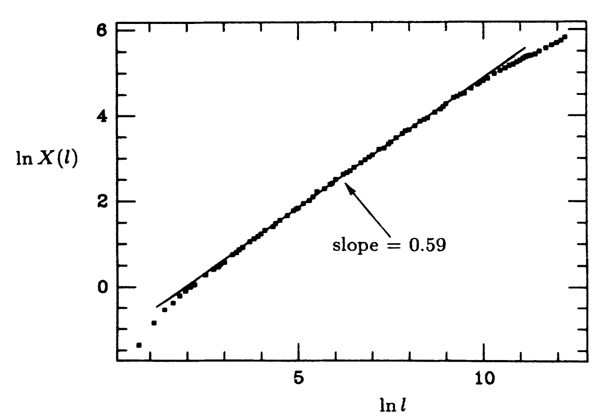

FIG. 2. Scaling of the standard deviation σ of the plots obtained by tracing five DLA clusters of 250 000 particles; $\sigma(l)$ was calculated by taking an average over the widths obtained for individual intervals of length l.

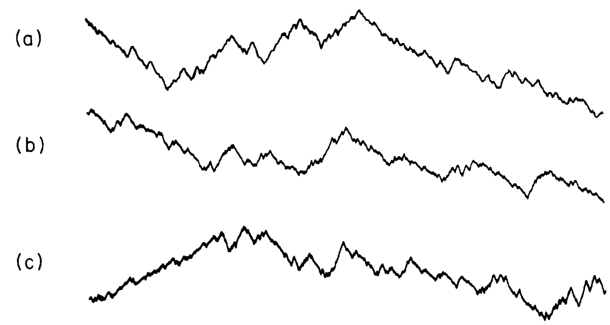

FIG. 3. Visual demonstration of the self-affinity of functions generated by tracing an aggregate. Plots obtained for (a) $l=1000$; (b) $l=4000$; (c) $l=16\,000$ steps have been rescaled onto the same interval by shrinking the horizontal and the vertical extension of the plots by the factors $\frac{1}{4}$ and $(\frac{1}{4})^{0.59}$, respectively. Self-affinity is indicated by the fact that the overall appearance of the resulting functions is the same (they are statistically identical).

III. SCALING APPROACH

The following scaling argument leads to a simple expression between the exponent H and the fractal dimension D of the DLA clusters. Let us consider a circle of radius R centered on the particle of a large aggregate, where the perimeter walk is started from. Because of the self-similarity of the aggregate, the number of steps $L(R)$ that are made by the walker before it reaches the circle scales as $L(R)\sim N(R)\sim R^D$, where $N(R)$ is the number of particles within the circle.

This means that the corresponding plot of $X(l)$ has a horizontal extension (number of steps) proportional to R^D, while at the same time its vertical size is proportional to the radius of the circle R. In other words, the height of the signal scales with its length L as $L^{1/D}$. From this and the definitions (2) or (3) we get

$$H=\frac{1}{D} . \quad (5)$$

This result is in full agreement with our numerical estimate for H. Expression (5) is also known to hold for plots of fractional Brownian motion, where it follows from the definition.[12]

Further evidence showing that (5) is satisfied for fractal clusters can be obtained by studying the exact mathematical fractal shown in Fig. 4. It is easy to see that this construction—which has the topology similar to that of a DLA aggregate—leads to an $X(l)$ plot that is self-affine. Figure 4(b) demonstrates that each time the horizontal scale is increased by a factor of 5, the vertical extension of the plot becomes three times larger. This behavior corresponds to $H=\ln3/\ln5$, and this means that for the fractal in Fig. 4(a) Eq. (5) is exactly satisfied, since its dimension is $D=\ln5/\ln3$. We have also numerically determined $\sigma(l)$ for this construction to see how fast is the convergence of H in the simulations to its true value. The results show that even for sizes corresponding to an aggregate of 3125 particles the observed scaling is ex-

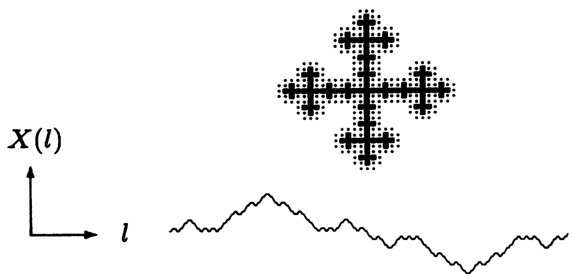

FIG. 4. This figure demonstrates that the relationship $H = 1/D$ is exactly satisfied on the deterministic fractal shown in (a).

tremely accurate, and the related slope deviates from the exact value by less than 3%.

IV. DISCUSSION

Our results for the sequence of X (or Y) coordinates obtained by tracing a diffusion-limited aggregate show that the associated plots can be considered as self-affine functions whose scaling behavior is determined by the exponent $H = 1/D$, where D is the fractal dimension of the aggregates. Accordingly, the local fractal dimension D_{SA} of these self-affine plots is related to D through the relation $D_{SA} = 2 - H = 2 - 1/D$.

Next we would like to point out how the kinds of plots studied in this work differ from those obtained by plotting the X coordinates of a fractional Brownian motion (FBM) in one dimension. In the case of FBM the increments of X represent a Gaussian random variable, and in analogy with (2) they satisfy the expression $\langle X^2(t) \rangle \sim t^{2H}$, where t denotes the time (corresponding to the number of steps for a perimeter walk). Although the width of the $X(l)$ curves scales with l as the width of the FDM plots for the same H, there are at least two important differences between these walks:

(i) The sequence of $X(l)$ and $Y(l)$ coordinates corresponding to two fractional Brownian walks in one dimension can be used to reconstruct an FBM in the plane. Obviously, the cluster obtained in this way statistically never produce a DLA cluster. From this, one can see that self-affine plots with the same scaling behavior can correspond to rather different actual functions. The situation is similar to the case of self-similar fractals: two objects with the same fractal dimension may have very different geometrical properties.

(ii) The scaling of $X(l)$ shown in Fig. 2 is far from being perfect. In fact, in a similar plot of an ordinary random walk, the data are much closer to a straight line. The deviation seen for the DLA case is likely to be relevant, since the simulations were carried out on a large scale.

The above observations raise the question of characterizing the difference between self-affine curves with the same H. For self-similar fractals such as DLA clusters, several quantities have been shown to be useful in describing the particular geometry of the aggregates. For example, it was found that the angular correlation function[13] and the three-point correlation function[14] provide further important information about the structure of DLA clusters.

The fact that the scaling of the width corresponding to the perimeter walk is systematically worse than for a standard random walk is likely to indicate the deviation of a diffusion-limited aggregate from a perfect isotropic fractal. This is in accordance with the findings reported in the above quoted works. Our results suggest that in the self-affine case, characteristics alternative to the exponent H should be studied as well. The investigation of some of these quantities is in progress.

ACKNOWLEDGMENTS

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[1] *Random Fluctuations and Pattern Growth,* edited by H. E. Stanley and N. Ostrowsky (Kluwer, Dordrecht, 1988).
[2] J. Feder, *Fractals* (Plenum, New York, 1988).
[3] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).
[4] T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).
[5] T. Vicsek, in *Random Fluctuations and Pattern Growth,* edited by H. E. Stanley and N. Ostrowsky (Kluwer, Dordrecht, 1988), p. 312; V. K. Horváth and T. Vicsek, J. Phys. A **22**, L259 (1990).
[6] M. Matsushita and S. Ouchi, Physica D **38**, 246 (1989).
[7] See, e.g., R. M. Ziff, Phys. Rev. Lett. **56**, 545 (1986), and references therein.
[8] S. Tolman and P. Meakin, Phys. Rev. A **40**, 428 (1989).
[9] F. Family and T. Vicsek, J. Phys. A **18**, 75 (1985).
[10] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[11] B. B. Mandelbrot, Physica Scripta **32**, 257 (1985); B. B. Mandelbrot, in *Fractals in Physics,* edited by L. Pietronero and E. Tosatti (Elsevier, Amsterdam, 1983), p. 3.
[12] R. F. Voss, in *The Science of Fractal Images,* edited by H. O. Peitgen and D. Saupe (Springer, Berlin, 1988), Chap. 1.
[13] P. Meakin and T. Vicsek, Phys. Rev. A **32**, 685 (1985); M. Kolb, J. Phys. Lett. **46**, 631 (1985).
[14] T. C. Halsey and P. Meakin, Phys. Rev. A **32**, 2546 (1985).