Universality Classes for Interface Growth with Quenched Disorder

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We present numerical evidence for the existence of two distinct universality classes characterizing driven interface roughening in the presence of quenched disorder. The evidence is based on the behavior of λ, the coefficient of the nonlinear term in the growth equation. Specifically, for three of the models studied, $\lambda \to \infty$ at the depinning transition, while for the two other models, $\lambda \to 0$.

The motion of a nonequilibrium interface in a disordered environment has attracted much attention. Fluid flow in a porous medium is a typical experimental realization of these phenomena, but applications range from wetting phenomena to the motion of flux lines in the presence of disorder [1]. The origin of different universality classes is well understood for growth in which thermal (or time-dependent) noise dominates the roughening process. However, recently several independent studies [1–8] have noted that *quenched* noise—which is independent of time and depends only on the position of the interface—may change the universality class, generating interfaces with anomalously large roughness exponents.

In the typical case, a d-dimensional interface characterized by a height $h(x,t)$ moves in a $(d+1)$-dimensional disordered medium. The randomness of the medium can be described by a quenched noise $\eta(x,h)$. In the presence of an external driving force F, the simplest growth equation describing the zero-temperature dynamics of the interface is [2]

$$\partial_t h = F + \nu \nabla^2 h + \eta(x,h). \tag{1}$$

The $\nu\nabla^2 h$ term mimics a surface tension and acts to *smooth* the interface, while the quenched noise $\eta(x,h)$ works to *roughen* the interface. It is generally assumed that the quenched noise has zero mean and is uncorrelated.

An interface characterized by (1) moves with a finite velocity v_0 if the driving force exceeds a critical value F_c, while for $F < F_c$ it is pinned by the disorder. When $F \to F_c$, one finds

$$v_0 \sim f^\theta, \tag{2}$$

where $f \equiv (F - F_c)/F_c$ is the reduced force and θ is the velocity exponent.

Recently, a number of analytical [2] and numerical [3–7] studies focused on understanding the nature of the depinning transition and obtaining accurate estimates for the critical exponents. Renormalization group (RG) studies [2] of Eq. (1) find a roughness exponent $\alpha = (4 - d)/3$, but a number of numerical models [3–7] revealed exponents whose values can be quite different from the RG predictions. Wetting fluid invasion models gave $\alpha \simeq 0.8$ for $1+1$ dimension, and investigation of the random field Ising model (RFIM) in $2+1$ dimension gave $\alpha \simeq 0.67$ [3]. Solid-on-solid type models gave $\alpha \simeq 0.63$

for $1 + 1$ dimension, and $\alpha \simeq 0.48$ for $2 + 1$ dimension [4,5], while a discretized solid-on-solid model of Eq. (1) gave $\alpha \simeq 1.25$ for $1+1$ dimension, and $\alpha \simeq 0.75$ for $2 + 1$ dimension [7]. A similar scattering is found for the values of the other exponents characterizing the depinning transition $(F = F_c)$.

Here we report simulations of five distinct models that have been introduced to investigate the motion of an interface in the presence of quenched disorder [3–7]. Our findings suggest the existence of two different universality classes. One universality class is described by the nonlinear growth equation [9]

$$\partial_t h = F + \nu \nabla^2 h + \lambda(\nabla h)^2 + \eta(x,h), \tag{3}$$

and we find that for models in this universality class

$$\lambda \sim f^{-\phi}. \tag{4}$$

The second universality class is described, at the depinning transition, by (1). We propose that the existence of the two universality classes is the origin of the systematic differences between the exponents predicted by RG calculations and estimates from numerical simulations. We find that measuring λ can give information about the universality classes to which a given growth process belongs.

To calculate λ, we follow Ref. [10] and impose a "tilt" of slope m on the interface. For a $(1+1)$-dimensional system, we consider a lattice with L columns, and "build in" the tilt by implementing helicoidal boundary conditions, $h(0,t) = h(L,t) - Lm$ and $h(L+1,t) = h(1,t)+Lm$. For a $(2 + 1)$-dimensional system, we use the same boundary conditions, so the tilt occurs only in one direction.

For a model described by Eq. (1), the local velocity v is *independent* of the tilt. However, if a nonlinear term is present in addition to the linear term, then from (3) it follows that [10]

$$v = v_0 + \lambda m^2. \tag{5}$$

Thus by varying the tilt m, we can test for the presence of nonlinear terms in the growth equation and calculate the coefficient λ.

First, we treat the model introduced in Ref. [4], for which it was shown, in $1+1$ dimension, that the interface at F_c is pinned by a directed percolation (DP) cluster [4,5], and that the critical dynamics are controlled by a divergent correlation length parallel to the interface

$\xi_\| \sim f^{-\nu_\|}$ with $\nu_\| \simeq 1.73$. This model, referred to as "DP-1," excludes overhangs, and gives rise to a self-affine interface at the depinning transition, with a roughness exponent $\alpha \simeq 0.63$ [11].

In the DP-1 model, in $1 + 1$ dimension, we start from a horizontal interface at the bottom edge of a lattice of size L. At every site of the lattice we define a random, uncorrelated quenched variable, the noise η, with magnitude in the range $[0,1]$. During the time evolution of the interface, we choose one of the L columns at random. If the difference in height to the lowest neighbor is larger than $(+1)$, this lowest neighboring column grows by one unit. Otherwise, the chosen column grows one unit provided the noise on the site above the interface is smaller than the driving force F. The unit time is defined to be L growth attempts.

We measure the velocity of the interface for different reduced forces f and different tilts m. The results for $1 + 1$ dimension are shown in Fig. 1(a). For a fixed force f, we find that the interface velocity changes with the tilt m, indicating the existence of nonlinear terms. Near the depinning transition $(f \to 0)$, the velocity curves become "steeper" and from (5), we infer that λ must increase.

To measure λ, we first attempt to fit a parabola with the tilt-dependent velocities in the vicinity of zero tilt. The calculations indicate that as we approach the depinning transition, λ diverges according to Eq. (4). However, in the vicinity of F_c, the velocity curves lose their parabolic shape for large tilts [see Figs. 1(a) and 2], indicating the presence of other terms not included in (5).

We can understand the breakdown of (5) for large m using scaling arguments. Substituting Eqs. (2) and (4) into (5), we find

$$v(m, f) \propto f^\theta + a f^{-\phi} m^2. \tag{6}$$

Equation (6) indicates that the velocity curves corresponding to two different forces f_1 and f_2, with $f_1 > f_2$, will intersect at a tilt $m_\times$ (see Fig. 2). For tilts greater than $m_\times$, $v(m, f_1) < v(m, f_2)$, a clearly unphysical prediction; the average velocity, for the same tilt, should be larger for the larger force. Thus the velocity cannot follow a parabola for arbitrarily large m, and a crossover to a different behavior than that of Eq. (6) must occur for values of the tilt larger than $m_\times$.

Letting $f_1 - f_2 \to 0$, we find from (6) that the crossing point of the two corresponding parabolas scales as

$$m_\times^2 \sim f^{\theta+\phi}. \tag{7}$$

Equations (6) and (7) motivate the scaling form for the velocities

$$v(m, f) \sim f^\theta g(m^2/f^{\theta+\phi}), \tag{8}$$

where $g(x) \sim$ const for $x \ll 1$, and $g(x) \sim x^{\theta/(\theta+\phi)}$ for $x \gg 1$ [12]. Figure 3(a) shows the data collapse we obtain using (8), and the results of Fig. 1(a) rescaled with exponents $\theta = 0.64 \pm 0.08$, $\phi = 0.64 \pm 0.08$ for $1 + 1$ dimension [13].

The scaling behavior (8) is not limited to the DP-1 model in $1 + 1$ dimension, for $2 + 1$ dimension and for the models introduced in Refs. [5,6] we find a very similar behavior. We refer to these models as "DP-2" [5] and
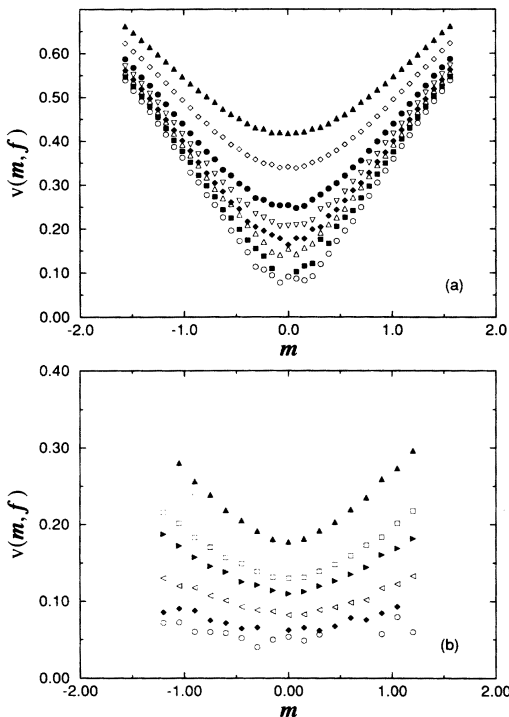
FIG. 1. Dependence on the tilt m of the average velocity, (a) in the DP-1 model and (b) in the RFIM. Data for different forces f are indicated by different symbols, ranging from 0.016 (bottom curve) to 0.350 (top curve) for the DP-1 model, and from 0.014 (bottom curve) to 0.143 (top curve) for the RFIM. In (a) we show velocities, for the DP-1 model, for a system of size 512 in 1+1 dimension. In (b) are plotted the velocities for the RFIM in the SA regime $(\Delta = 3)$, for a $(2+1)$-dimensional system of size 40×40.
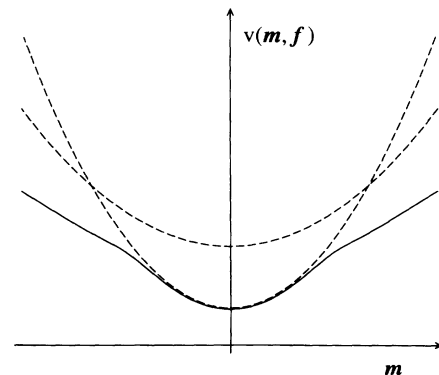

FIG. 2. Here we exemplify the "noncrossing" effect on the velocity parabolas. We show a perfect parabolic behavior for two different forces, $f_1 > f_2$ (dashed lines) as predicted by Eq. (5). Also shown is the "curving back" of the velocity curve for the smaller force f_2 (solid line) in order not to cross the velocity curve for f_1.
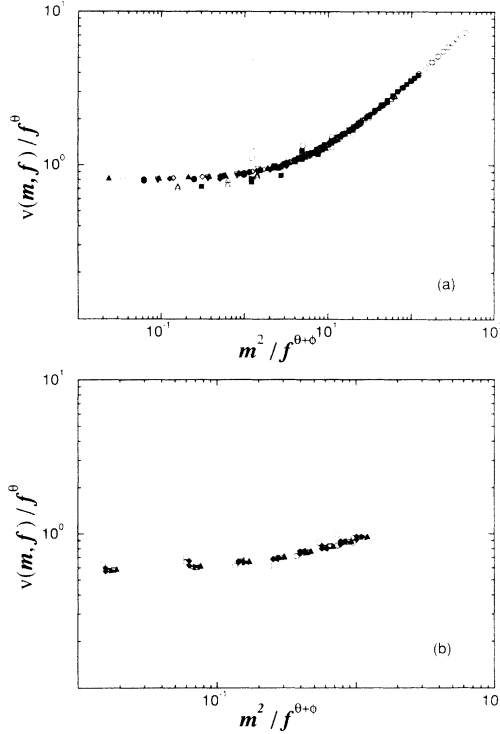
FIG. 3. Data collapse according to (8), using the same symbols for the velocities shown in Fig. 1. In (a) we present the rescaled results for the DP-1 model in $1 + 1$ dimension and in (b) the rescaling of the $(2 + 1)$-dimensional results for the RFIM, in the SA regime ($\Delta = 3.0$).

TABLE I. Exponents for the five studied models (see definitions in the text). A negative value of ϕ means that $\lambda \to 0$ when $f \to 0$. We argue in the text that the models above the horizontal line (DP-1, DP-2, and DP-3) belong to the universality class of Eq. (3) and can be mapped, in $1 + 1$ dimension, to DP. The models below the line belong to the universality class of Eq. (1). See [14] for a discussion on the values of the exponents for the DP-2 model.

Model		$1 + 1$ dimension		$2 + 1$ dimension	
		θ	ϕ	θ	ϕ
DP-1		0.64 ± 0.08	0.64 ± 0.08	0.80 ± 0.12	0.30 ± 0.12
DP-2		0.59 ± 0.12	0.55 ± 0.12		
DP-3		0.70 ± 0.12	0.65 ± 0.12		
RFIM	SA	$\cdots$	$\cdots$	0.60 ± 0.11	-0.70 ± 0.11
	SS	0.31 ± 0.08	-0.65 ± 0.13		
SOS-1		0.26 ± 0.07	$\cdots$		

"DP-3" [6]. We simulated them in $1 + 1$ dimension, and were able to rescale the velocities according to (8) using the exponents presented in Table I [14].

Another model we studied was the RFIM, which allows for overhangs; and for certain values of its parameters can be mapped to percolation [3]. In the RFIM, spins on a square lattice interact through the Hamiltonian

$$\mathcal{H} \equiv - \sum_{\langle i,j \rangle} S_i S_j - \sum_i [F + \eta(i, h)] S_i, \qquad (9)$$

where $S_i = \pm 1$, F now denotes the external magnetic field, and η is the time-independent local random field (i.e., quenched noise) whose values are uniformly distributed in the interval $[-\Delta, \Delta]$. The strength of the quenched disorder is characterized by the parameter Δ. At time zero, all spins are "down"—except those in the first row, which are initially up. During the time evolution of the system, we flip any down spin that is "unstable," i.e., whenever the flip will lower the total energy of the system. The control parameter of the depinning transition is the external magnetic field F; the unit time corresponds to flipping all unstable spins [15].

For dimension $1 + 1$, there are two morphologically different regimes, depending on the strength Δ of the disorder (i.e., of the random fields). For $\Delta > 1.0$, the interface is self-similar (SS), while for $\Delta < 1.0$ it is faceted (FA). For dimension $2 + 1$, there is again a FA regime ($\Delta < 2.4$),

a SS regime ($\Delta > 3.4$), and also a self-affine (SA) regime in between ($2.4 < \Delta < 3.4$) [3]. The SA regime, which exists only for $2 + 1$ dimension, is the only regime of the RFIM for which either Eq. (1) or (3) could apply. In the SS regime, the interface is not single valued, while in the FA regime, lattice effects dominate the growth.

Our results show that, for the FA regime, the RFIM behaves in a similar fashion to the other three models, in that the coefficient of the nonlinear term diverges at the depinning transition. However, although (8) is still valid for the SA and SS regimes, we find a negative ϕ, thus $\lambda \to 0$. This behavior can be understood, for the SS regime, by considering that near the depinning transition the morphology of the interface corresponds to the hull of a percolation cluster, which has no well-defined orientation [3]. Thus a change in the boundary conditions will not affect the growth process, and we cannot expect any divergence of a possible nonlinear term when the magnetic field approaches its critical value. On the other hand, for large fields, the effect of the quenched disorder diminishes, and we can observe an average interface orientation. For such values of field, we expect the presence of nonlinear terms to be felt. Although for the SA regime the behavior of λ is similar [see Figs. 1(b) and 3(b)], the reasons so far cannot be understood.

These results lead us to conclude that in the SA regime the RFIM belongs to the universality class of Eq. (1). This conclusion is further supported by the agreement between the numerically determined exponents, $\alpha \simeq 0.67$ and $\theta \simeq 0.60$ for $2+1$ dimension, and the RG predictions, $\alpha = \theta = 2/3$ [2].

Finally we studied the discretized solid-on-solid version of Eq. (1), referred to as "SOS-1" [7], and find that for any reduced force, $\lambda = 0$.

The results of Table I show, for $1 + 1$ dimension, a separation into two groups in the values of the critical exponents for the five models studied [14]. In the following we argue that this separation reflects the existence of two distinct universality classes, described by the two continuum growth equations, (1) and (3). For the SOS-1

model and for the RFIM, in the SA regime, we find that λ either is zero or goes to zero at the depinning transition. Thus the scaling behavior of these models should be correctly described by (1). For the DP-1, DP-2, and DP-3 models we observe a divergent λ, indicating that nonlinearities are relevant near the depinning transition. Thus to properly describe the scaling properties of these models it is necessary to study (3), since (1) does not include the nonlinear term $\lambda(\nabla h)^2$. Further evidence of the existence of the two universality classes is given by the values of roughness exponents. The models for which λ diverges at the depinning transition [4–6], predict $\alpha \simeq 0.63$, in agreement with the mapping to DP [16]. On the other hand, models in the universality class of Eq. (1) [3,7], gave roughness exponents typically larger, in better agreement with the RG predictions [2].

Finally, we propose the study of the behavior of λ at the depinning transition as a general method for identifying the universality class of a given growth process in disordered media. This method has been successfully applied to the motion of an elastic string in a random medium, suggesting that the nonlinear term might be generated by the anisotropy in the disorder [17].

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[9] Equation (3) is similar to the Kardar-Parisi-Zhang equation [M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986)] with the difference that here we have quenched noise instead of thermal disorder.

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[11] The mapping to DP allows us to calculate the roughness exponent α at the depinning transition [4,5]. Near the transition the width of the interface w is given by the perpendicular correlation length $\xi_\perp$, and the parallel correlation length $\xi_\parallel$ is of the order of the system size. So $w \sim L^{\nu_\perp/\nu_\parallel}$, giving for the roughness exponent $\alpha = \nu_\perp/\nu_\parallel \simeq 0.63$.

[12] The consistency of the argument developed in the text can be verified by expanding (8) to first order, whereupon we recover (6).

[13] The value of the velocity exponent can also be obtained by considering the mapping to DP [4]. The average velocity close to the depinning transition scales as $\xi_\perp/\xi_\parallel$, giving $\theta = \nu_\parallel - \nu_\perp \simeq 0.64$.

[14] As was shown in Refs. [4] and [5], the DP-1 and DP-2 models belong to the same universality class, with exponents $\alpha \simeq 0.63$ and $\theta \simeq 0.64$ [11,13]. The deviation of the numerical values in the exponents for the DP-2 model can be explained by the slow convergence of this model to the asymptotic scaling regime. The large error bars for all models arise from the fact that simulations were done for relatively small systems. Despite the differences in the values of the scaling exponents, based on the numerical results combined with the exact results of Refs. [4] and [5], we conclude that the models, DP-1, DP-2, and DP-3, share the same set of scaling exponents, thus belong to the same universality class.

[15] As was noted by Koiller *et al.* [3] different methods of defining time in the RFIM lead to different values of the exponents. Here we applied one of the methods suggested in [3], in which the unit time corresponds to flip all unstable interface spins. The value of the exponent given in [3] was obtained assuming that the time it takes a spin to flip depends on the energy gain, for the system, in the flipping.

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