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Deterministic scale-free networks

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Abstract

Scale-free networks are abundant in nature and society, describing such diverse systems as the world wide web, the web of human sexual contacts, or the chemical network of a cell. All models used to generate a scale-free topology are stochastic, that is they create networks in which the nodes appear to be randomly connected to each other. Here we propose a simple model that generates scale-free networks in a deterministic fashion. We solve exactly the model, showing that the tail of the degree distribution follows a power law. © 2001 Published by Elsevier Science B.V.

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Networks are ubiquitous in nature and society [1–3], describing various complex systems, such as the society, a network of individuals linked by various social links [4–7]; the Internet, a network of routers connected by various physical connections [8,9]; the world wide web, a virtual web of documents connected by uniform resource connectors [10] or the cell, a network of substrates connected by chemical reactions [11,12]. Despite their diversity, most networks appearing in nature follow universal organizing principles. In particular, it has been found that many networks of scientific interest are scale-free [13,14], that is, the probability that a randomly selected node has exactly k links decays as a power law, following $P(k) \sim k^{-\gamma}$, where γ is the degree exponent. The list of documented scale-free networks now include the world wide web [10,15], the Internet [8], the cell [11,12], the web of human sexual contacts [16], the

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language [17], or the web of actors in Hollywood [18], most of which appear to have degree exponents between two and three.

The high interest in understanding the topology of complex networks has resulted in the development of a considerable number of network models [13,14,19–27]. Most of these are based on two mechanisms: incremental growth and preferential attachment [13,14]. Incremental growth captures the fact that networks are assembled through the addition of new nodes to the system, while preferential attachment encodes the hypothesis that new nodes connect with higher probability to more connected nodes. Both of these mechanisms have been supported by extensive empirical measurements [6,28,29], indicating that they are simultaneously present in many systems with scale-free network topology.

Stochasticity is a common feature of all network models that generate scale-free topologies. That is, new nodes connect using a probabilistic rule to the nodes already present in the system. This randomness present in the models, while in line with the major features of networks seen in nature, makes it harder to gain a visual understanding of what makes them scale-free, and how do different nodes relate to each other. It would therefore be of major theoretical interest to construct models that lead to scale-free networks in a deterministic fashion. Here we present such a simple model, generating a deterministic scale-free network using a hierarchical construction.

1. Model description

The construction of the model, that follows a hierarchical rule commonly used in deterministic fractals [30,31], is shown in Fig. 1. The network is built in an iterative fashion, each iteration repeating and reusing the elements generated in the previous steps as follows:

Step 0: We start from a single node, that we designate as the *root* of the graph.

Step 1: We add two more nodes, and connect each of them to the root.

Step 2: We add two units of three nodes, each unit identical to the network created in the previous iteration (step 1), and we connect each of the *bottom* nodes (see Fig. 1) of these two units to the root. That is, the root will gain four more new links.

Step 3: We add two units of nine nodes each, identical to the units generated in the previous iteration, and connect all eight bottom nodes of the two new units to the root.

These rules can be easily generalized. Indeed, step n would involve the following operation:

Step n: Add two units of 3^{n-1} nodes each, identical to the network created in the previous iteration (step $n-1$), and connect each of the 2^n bottom nodes of these two units to the root of the network.

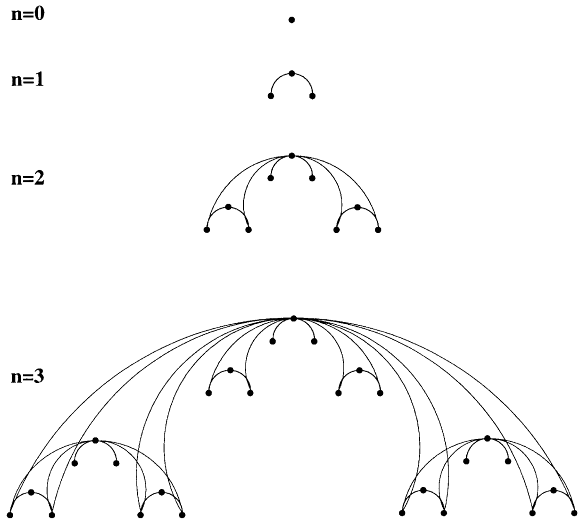

Fig. 1. Construction of the deterministic scale-free network, showing the first four steps of the iterative process.

2. Analytical solution

Thanks to its deterministic and discrete nature, the model described above can be solved exactly. To show its scale-free nature, in the following we concentrate on the degree distribution, $P(k)$.

The tail of the degree distribution is determined by the most connected nodes, or hubs. Clearly the biggest hub is the root, and the next two hubs are the roots of the two units added to the network in the last step. Therefore, in order to capture the tail of the distribution, it is sufficient to focus on the hubs.

In step i the degree of the most connected hub, the root, is $2^{i+1}-2$. In the next iteration two copies of this hub will appear in the two newly added units. As we iterate further, in the nth step 3^{n-i} copies of this hub will be present in the network. However, the two newly created copies will not increase their degree after further iterations. Therefore, after n iterations there are $(2/3)3^{n-i}$ nodes with degree $2^{i+1}-2$. From this we obtain that the tail of the degree distribution, determined by the hubs, follows

$$P(k) \sim k^{-\frac{\ln 3}{\ln 2}}.$$

Thus the degree exponent is

$$\gamma = \frac{\ln 3}{\ln 2}.$$

The origin of this scaling behavior can be understood by inspecting the model's construction. Indeed, at any moment we have a hierarchy of hubs, highly connected nodes which are a common component of scale-free networks. The root is always the largest hub. However, at any step there are two hubs whose connectivity is roughly a half of the root's connectivity, corresponding to the roots of the two units added at step $n-1$. There are six even smaller hubs, with connectivity $2^{n-1}-2$, corresponding to the root of the units added at time $n-2$, and so on. This hierarchy of hubs is responsible for the network's scale-free topology. As the number of hubs increases as powers of 3, while the number of links only as powers of two, the degree exponent is expected to be a simple multiple of $\ln 3/\ln 2$.

3. Discussion

The model introduced above offers a deterministic construction of a scale-free network. One of its interesting properties is its apparent self-similarity. On the other hand, while the networks generated at step n and $n-1$ have an identical large-scale topology, self-similarity is not complete. The special role played by the root, the major hub in the network, violates complete self-similarity. Indeed, the root keeps a detailed record of the system size through the number of links it has. Such global information is never present in a local element of a fractal, the common example of a self-similar object. This unique feature of our construction reflects a common property of all scale-free networks, rooted in the nonlocal growth rule that generates scale-free systems. Indeed, in stochastic models the probability that a node connects to an existing node in the system contains implicitly the information about the whole system. While the connectivity of the hub in a stochastic network is not equal to the system size, it typically varies as a simple function of N. Similarly, here the largest hub, the root of our construction, keeps track of the system size in a trivial way, as 2/3 of the nodes are linked to it.

The proposed model generates a network with a fixed $\gamma = \ln 3/\ln 2$ degree exponent. However, one can easily modify the model to change the scaling exponent by varying the number of links connected to the root at each step. For example, if in each iteration we connect *all* nodes of the two new units to the root (and not only the 2^n base nodes), the degree exponent changes to $\gamma = \ln 3/\ln 2$. Similarly, by definition the model discussed here has a zero clustering coefficient [6], as it does not generate triangles of connected nodes. It is easy to change the rules, without changing the scaling exponent, to obtain a network that displays nonzero clustering. Indeed, the version in which all nodes are linked to the root has a finite clustering coefficient. These and further variants of the model will be discussed in forthcoming publications.

A crucial unique property of the model is its hierarchical structure. It is not clear whether the hierarchy seen in this model is a unique property of its construction, or it is an intrinsic property of all scale-free networks. Uncovering the elements of the hierarchy could be an important future task in our understanding of scale-free networks. The present model explicitly builds such a hierarchical construction into the network, as at each iteration we connect nodes to each other in a hierarchical fashion.

In conclusion, we introduced a simple model that allows us to construct a deterministic scale-free network. We have shown that the scaling exponent characterizing the tail of the $P(k)$ distribution can be calculated analytically. An important feature of the model is its hierarchical topology, as at each iteration it combines identical elements to generate a larger network. The method that lead to the construction of the network naturally lends itself to immediate generalizations, leading to structures that are similar in spirit, but rather different in detail, generating networks with different scaling exponents and connectivity.

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